Linear Algebra Review

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Basic Notation:

- $A \in \mathbb{R}^{m \times n}$ denotes a matrix with m rows and n columns where the entries of A are real numbers.

- $x \in R^n$ denotes a vector with n entries. Usually, a vector x will denote a column vector, a matrix with n rows and 1 column. We typically write x^T to represent a row vector, a matrix with 1 row and n columns.

I.e. $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ ← Column vector

$x^T = [x_1, x_2, \ldots, x_n]$ ← Row Vector

- x_i denotes the i^{th} element of vector x.

- A_{ij} / a_{ij} denotes the element in matrix A at the i^{th} row and j^{th} column.

- $a_j / A_{:,j}$ denotes the j^{th} column in matrix A.

- $a_i^T / A_{i,:}$ denotes the i^{th} row in matrix A.

Identity Matrix:

- The identity matrix, denoted $I \in R^{n \times n}$, is a square matrix with 1's on the main diagonal and 0's elsewhere.
- It has the property that for all $A \in R^{m \times n}$, $AI = A = IA$, where the size of I is determined by the dimensions of A so matrix multiplication is possible.
- E.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3 by 3 identity matrix.

Diagonal Matrix:

- A diagonal matrix is a matrix where all elements not on the main diagonal are 0.
- We typically denote a diagonal matrix using $diag(d_1, d_2, \ldots, d_n)$.
- The identity matrix, I, is equal to $diag(1, 1, \ldots, 1)$.
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  are all diagonal matrices.

  Note: There is no restriction that elements along the main diagonal can't be 0.

Square Matrix:

- A square matrix is a matrix with the same number of rows and columns.
- E.g. $\begin{bmatrix} 1 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 6 \\ 0 & 9 & 7 \\ 3 & 5 & 4 \end{bmatrix}$ are all square matrices.

Main Diagonal:

- The main diagonal of matrix A is the list of entries A_{ij} where $i=j$.
- E.g. $\begin{bmatrix} 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 \\ 1 & 6 \\ 3 & 2 \end{bmatrix}$, $\begin{bmatrix} 7 & 1 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

  All elements in green are part of their array's main diagonal.

Transpose:

- The transpose of a matrix, denoted as A^T, results from flipping the rows and cols.

- Let $A \in R^{m \times n}$. Then, $A^T \in R^{n \times m}$.

- $(A^T)_{ji} = A_{ij}$

- E.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

- Properties:
  1. $(A^T)^T = A$
  2. $(AB)^T = B^T A^T$
  3. $(A+B)^T = A^T + B^T$

Symmetric Matrices:

- A square matrix is symmetric if that matrix is equal to its transpose.
  I.e. $A \in R^{n \times n}$ is symmetric if $A = A^T$.

- A square matrix is anti-symmetric if that matrix is equal to the negative of its transpose.
  I.e. $A \in R^{n \times n}$ is anti-symmetric if $A = -A^T$.

- E.g. $\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}$ is symmetric

  $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ is anti-symmetric

- For any square matrix $A \in R^{n \times n}$, $A + A^T$ is symmetric and $A - A^T$ is anti-symmetric.
  Hence, we can write any square matrix $A \in R^{n \times n}$ as
  $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

The Trace

- The trace of a square matrix $A \in R^{n \times n}$, denoted as $tr(A)$ / trA is the sum of the elements along the main diagonal.

$$tr(A) = \sum_{i=1}^{n} A_{ii}$$

- Properties:
  1. For $A \in R^{n \times n}$, $tr(A) = tr(A^T)$
  2. For $A, B \in R^{n \times n}$, $tr(A+B) = tr(A) + tr(B)$
  3. For $A \in R^{n \times n}$, $t \in R$, $tr(tA) = t \cdot tr(A)$
  4. For A, B s.t. $AB \in R^{n \times n}$, $tr(AB) = tr(BA)$
  5. For A, B, C s.t. $ABC \in R^{n \times n}$, $tr(ABC) = tr(BAC) = tr(CAB) = \ldots$

Norm:

- The norm of a vector, denoted as $\|x\|$, informally is the measure of the length of the vector.

  More formally, a norm is any function $f: R^n \to R$ that satisfies 4 properties:
  1. $\forall x \in R^n$, $f(x) > 0$
  2. $f(x) = 0$ iff $x = 0$
  3. $\forall x \in R^n$, $t \in R$, $f(tx) = |t| f(x)$
  4. $\forall x, y \in R^n$, $f(x+y) \leq f(x) + f(y)$

- Let $p \geq 1$ be a real number. The p-norm/ℓp-norm of vector x is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

For $p=1$, we get the taxicab norm.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

For $p=2$, we get the Euclidean norm.

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

Note: $(\|x\|_2)^2 = x^T x$

For $p=\infty$, we get the infinity norm.

$$\|x\|_\infty = \max_i |x_i|$$
$$= \max(|x_1|, |x_2|, \dots, |x_n|)$$

- Norms can also be defined for matrices, such as the Frobenius Norm.

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (A_{ij})^2} \;\Bigg\}\; \text{The square root of the sum of the abs squares of its elements.}$$

$$= \sqrt{tr(A^T A)}$$

Linear Independence and Rank:

- A set of vectors $\{x_1, x_2, \ldots, x_n\}$ is linearly independent if no vector can be represented as a linear combination of the remaining vectors. Conversely, a vector that can be represented as a linear combination of the remaining vectors is linearly dependent.

  Another way to think about this is a set of vectors is linearly dependent if there is a non-trivial linear combination of the vectors that equal the zero vector. If no such linear combination exists, then the vectors are linearly independent.

- E.g. Are the vectors $V_1 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

  linearly dep or indep?

  Soln:

$$\begin{bmatrix} 2 & 1 & 4 \\ 5 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 - R_3$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 5 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$

Note: A set of vectors is linearly indep if you only get the trivial soln when you try to find coefficients s.t.
$a_1 \vec{v_1} + a_2 \vec{v_2} + \ldots + a_n \vec{v_n} = \vec{0}$

$R_2 - R_3$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right]$$

$R_3 + 3 \cdot R_1$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 1 & 12 & 0 \end{array}\right]$$

$R_2 / -2$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 12 & 0 \end{array}\right]$$

$R_2 - R_1$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 1 & 12 & 0 \end{array}\right]$$

$R_2 / -3$

$$\left[\begin{array}{ccc|c} -1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 12 & 0 \end{array}\right]$$

$R_1 \cdot (-1)$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 12 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 12 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since we get the trivial soln,
the 3 vectors are linearly independent.

- E.g. Are the vectors $V_1 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$V_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$?

Soln:

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right]$$

$R_3 - R_1$

$$\left[\begin{array}{ccc|c} 4 & -3 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -6 & 4 & 0 & 0 \end{array}\right]$$

$R_1 \cdot 2 + R_2$

$$\left[\begin{array}{ccc|c} 9 & -6 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ -6 & 4 & 0 & 0 \end{array}\right]$$

$R_1 / -3$

$$\left[\begin{array}{ccc|c} -3 & 2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ -6 & 4 & 0 & 0 \end{array}\right]$$

$R_3/2$

These 2 rows are the same.

$$\left[\begin{array}{ccc|c} -3 & 2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ -3 & 2 & 0 & 0 \end{array}\right]$$

$R_1 - R_3$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ -3 & 2 & 0 & 0 \end{array}\right] \leftarrow$$

Because there's a column without a pivot, the vectors are lin dep.

- The column rank of matrix A is the largest number of cols of A that constitute a linearly indep set.

- The row rank of a matrix A is the largest number of rows of A that constitute a linearly indep set.

  Note: For any matrix A, columnrank(A) = rowrank(A). As a result, this quantity is simply referred to as the rank of A, denoted as rank(A).

- Properties of rank:
  1. For $A \in R^{m \times n}$, $\text{rank}(A) \leq \min(m, n)$. If $\text{rank}(A) = \min(m,n)$, then A is said to be full rank.
  2. For $A \in R^{m \times n}$, $\text{rank}(A) = \text{rank}(A^T)$.
  3. For $A \in R^{m \times n}$, $B \in R^{n \times p}$, $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
  4. For $A, B \in R^{m \times n}$, $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

Inverse Matrix:

- The inverse of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted as A^{-1}, is the unique matrix s.t.

$$A^{-1}A = AA^{-1} = I$$

- Not all square matrices have an inverse. In this case, we say that the matrix is non-invertible or singular. If the matrix does have an inverse, we say that the matrix is invertible or non-singular.

- It is possible to show that A^{-1} exists iff A is full rank.

- Properties (Assume A and B are square and non-singular):
  1. $(A^{-1})^{-1} = A$
  2. If $Ax = b$, we can multiply by A^{-1} on both sides to get $x = A^{-1}b$.
  3. $(AB)^{-1} = B^{-1}A^{-1}$
  4. $(A^{-1})^T = (A^T)^{-1}$. For this reason, this matrix is often denoted as A^{-T}.

Orthogonal Matrices:

- 2 vectors are orthogonal if they are perpendicular to each other. (That means their dot product is 0.)

- E.g. Let $A = (1, 0, -1)$ and $B = (-1, 0, -1)$

$$A \cdot B = (1)(-1) + (0)(0) + (-1)(-1)$$
$$= -1 + 1$$
$$= 0$$

$\therefore$ A and B are orthogonal.

- A set of vectors, $\{V_1, V_2, \dots, V_n\}$ are mutually orthogonal if every pair of vectors are orthogonal.

  I.e. $V_i \cdot V_j = 0 \quad \forall i \neq j$

- E.g. Let $A = (1, 0, -1)$, $B = (1, \sqrt{2}, 1)$, $C = (1, -\sqrt{2}, 1)$

$$\begin{aligned} A \cdot B &= (1)(1) + (0)(\sqrt{2}) + (-1)(1) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} A \cdot C &= (1)(1) + (0)(-\sqrt{2}) + (-1)(1) \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} B \cdot C &= (1)(1) + (\sqrt{2})(-\sqrt{2}) + (1)(1) \\ &= 1 - 2 + 1 \\ &= 0 \end{aligned}$$

$\therefore$ $\{A, B, C\}$ are mutually orthogonal.

- A vector, x, is normalized if $\|x\|_2 = 1$.

- E.g. Let $A = (1, 0, -1)$

$$\begin{aligned} \|A\|_2 &= \sqrt{(1)^2 + (0)^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$\therefore$ A is not normalized.

- To normalize a vector, we just divide the vector by its magnitude.

- E.g. Let $A = (1, 0, -1)$.

  We've previously show that A is not normalized.

  Let $B = \frac{A}{\|A\|_2}$

  $B = \frac{1}{\sqrt{2}}(1, 0, -1)$

  $= (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$

  $\|B\|_2 = \sqrt{(\frac{1}{\sqrt{2}})^2 + (0)^2 + (-\frac{1}{\sqrt{2}})^2}$

  $= \sqrt{\frac{1}{2} + 0 + \frac{1}{2}}$

  $= \sqrt{1}$

  $= 1$

  $\therefore$ B is normalized.

- A set of vectors is orthonormal if every vector in the set is normalized and the set of vectors are mutually orthogonal.

- A square matrix is orthogonal if all of its cols are orthogonal to each other.

- The cols of a square matrix is orthonormal if all of its cols are normalized and it is orthogonal.

- Let U be a square, orthogonal matrix. Then:
  1. $U^T U = I = UU^T$
  2. $U^T = U^{-1}$

- Given a square, orthogonal matrix U and a vector x, we have the following property:

$$||Ux||_2 = ||x||_2$$

Note: The dimensions of U and x must be compatible.

Range and Nullspace:

- The span of a set of vectors is the set of all vectors that can be expressed as a linear comb of it.

  I.e. The span of a set of vectors, S, is denoted as $\text{span}(S)$ and is the set of all linear comb of these vectors.

  I.e. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of vectors

$$\text{span}(\{x_1, \ldots, x_n\}) = \left\{ v : v = \sum_{i=1}^{n} a_i x_i, \ a_i \in R \right\}$$

- If $\{x_1, \ldots, x_n\}$ is a set of n linearly indep vectors where each $x_i \in R^n$, then $\text{span}(\{x_1, \ldots, x_n\}) = R^n$

- E.g. $\text{span}(\{(1,0) \text{ and } (0,1)\}) = R^2$

- The projection of a vector $y \in R^m$ onto the span of $\{x_1, \ldots, x_n\}$ where each $x_i \in R^m$ is the vector $v \in \text{span}(\{x_1, \ldots, x_n\})$ s.t. v is as close as possible to y.

I.e. $\text{Proj}(y; \{x_1, \ldots x_n\}) = \text{argmin}_{v \in \text{span}(\{x_1, \ldots x_n\})} \|y - v\|_2$

Let $S = \{x_1, \ldots, x_n\}$. If S is mutually orthogonal, then we can find the projection of vector v this way:

$$\text{Proj}_w v = \frac{\overset{\text{Dot product}}{v \cdot x_1}}{x_1 \cdot x_1} x_1 + \frac{v \cdot x_2}{x_2 \cdot x_2} x_2 + \cdots + \frac{v \cdot x_n}{x_n \cdot x_n} x_n$$

where $w = \text{span}(\{x_1, \ldots, x_n\})$.

E.g. Find the projection of $(7, 2, 2, 1)$ on $\text{span}(\{(1, 2, 2, 4), (4, -2, 8, -4)\})$.

Soln:

$$(1, 2, 2, 4) \cdot (4, -2, 8, -4) = 4 + (-4) + 16 + (-16)$$
$$= 0$$

$$\frac{(7,2,2,1) \cdot (1,2,2,4)}{(1,2,2,4) \cdot (1,2,2,4)} (1,2,2,4) = \frac{7+4+4+4}{1+4+4+16} (1,2,2,4)$$
$$= \frac{19}{25} (1,2,2,4)$$

$$\frac{(7,2,2,1) \cdot (4,-2,8,-4)}{(4,-2,8,-4) \cdot (4,-2,8,-4)} (4,-2,8,-4) = \frac{28-4+16-4}{16+4+64+16} (4,-2,8,-4)$$
$$= \frac{36}{100} (4,-2,8,-4)$$

$$= \frac{19}{25}\begin{pmatrix}1\\2\\2\\4\end{pmatrix} + \frac{9}{25}\begin{pmatrix}4\\-2\\8\\-4\end{pmatrix}$$

$$= \begin{pmatrix}\frac{19}{25}\\ \frac{38}{25}\\ \frac{38}{25}\\ \frac{76}{25}\end{pmatrix} + \begin{pmatrix}\frac{36}{25}\\ \frac{-18}{25}\\ \frac{72}{25}\\ \frac{-36}{25}\end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix}55\\20\\110\\40\end{pmatrix}$$

E.g. Find the projection of $(2, 9, -4)$ on $\text{span}(\{(1,2,2), (2,1,-2)\})$.

Soln:

$$(1, 2, 2)\cdot(2, 1, -2) = 2+2-4$$
$$= 0$$

$$\frac{(2, 9, -4)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)}(1,2,2) = \frac{2+18-8}{1+4+4}(1,2,2)$$
$$= \frac{12}{9}(1,2,2)$$

$$\frac{(2,9,-4)\cdot(2,1,-2)}{(2,1,-2)\cdot(2,1,-2)}(2,1,-2) = \frac{4+9+8}{4+1+4}(2,1,-2)$$
$$= \frac{21}{9}(2,1,-2)$$

$$= \frac{\cancel{12}^{\,4}}{\cancel{9}_{\,3}}\begin{pmatrix}1\\2\\2\end{pmatrix} + \frac{\cancel{21}^{\,7}}{\cancel{9}_{\,3}}\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}4+14\\8+7\\8-14\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}18\\15\\-6\end{pmatrix} = \begin{pmatrix}6\\5\\-2\end{pmatrix}$$

If S is not mutually orthogonal, then we need to Gram-Schmidt process to make S mutually orthogonal.

Gram-Schmidt process:
Let $x_1, x_2, \dots, x_n$ be a set of vectors.

$$u_1 = x_1$$

$$u_2 = x_2 - \text{proj}_{u_1} x_2$$

$$u_3 = x_3 - \text{proj}_{u_1} x_3 - \text{proj}_{u_2} x_3$$

$$\vdots$$

$$u_k = x_k - \sum_{j=1}^{k-1} \text{proj}_{u_j} x_k$$

$u_1, u_2, \dots, u_k$ is the set of mutually orthogonal vectors.

E.g. Project $(4, 9)$ on $\text{span}(\{(3,1), (2,2)\})$.

Soln:

$$(3,1)\cdot(2,2) = 6+2 = 8$$

Need to use Gram-Schmidt

$$u_1 = (3,1)$$

$$u_2 = (2,2) - \text{Proj}_{u_1}(2,2)$$

$$= (2,2) - \frac{(3,1)\cdot(2,2)}{(3,1)\cdot(3,1)}(3,1)$$

$$= (2,2) - \frac{8}{10}(3,1)$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 12/5 \\ 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$$

$$(3,1)\cdot(-2/5, 6/5) = -6/5 + 6/5 = 0$$

$$\frac{(4,9)\cdot(3,1)}{(3,1)\cdot(3,1)}(3,1) = \frac{21}{10}(3,1)$$

$$\frac{(4,9)\cdot(-2/5, 6/5)}{(-2/5, 6/5)\cdot(-2/5, 6/5)}\left(\frac{-2}{5}, \frac{6}{5}\right) = \frac{23}{4}\left(-\frac{2}{5}, \frac{6}{5}\right)$$

Final soln: $\frac{21}{10}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{23}{4}\begin{pmatrix} -\frac{2}{5} \\ 6/5 \end{pmatrix}$

- The range/columnspace of a matrix $A \in \mathbb{R}^{m \times n}$, denoted as $R(A)$, is the span of the cols of A.

$$R(A) = \{ v \in \mathbb{R}^m : v = Ax, x \in \mathbb{R}^n \}$$

E.g. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

$$R(A) = \left\{ \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{x} \right\}$$

- The nullspace of a matrix $A \in \mathbb{R}^{m \times n}$, denoted as $N(A)$ is the set of all vectors that equal 0 when multiplied by A.

I.e. $N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

Note: The nullspace is always non-empty since it always contains the 0 vector.

- $R(A^T)$ and $N(A)$ are disjoint subsets that together span the entire space of $\mathbb{R}^n$. Sets of this type are called orthogonal complements and are denoted as $R(A^T) = N(A)^{\perp}$.

Determinant:

The

- The determinant of a square matrix, denoted as $|A|$ or $\det A$, is the following:

$$A = \begin{bmatrix} a_1 \end{bmatrix} \text{ (1 by 1 matrix)}$$
$$|A| = a_1$$

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \text{ (2 by 2 matrix)}$$
$$|A| = a_1 \cdot a_4 - a_2 \cdot a_3$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$$
$$|A| = a_1 (a_5 \cdot a_9 - a_6 \cdot a_8) - a_2 (a_4 \cdot a_9 - a_6 \cdot a_7) + a_3 (a_4 \cdot a_8 - a_5 \cdot a_9)$$

- Properties:
  1. $|I| = 1$
  2. $|tA| = t|A|$
  3. If we swap any 2 rows, then the det is $-|A|$.
  4. $|A| = |A^T|$
  5. $|AB| = |A||B|$
  6. $|A| = 0$ iff A is singular (non-invertible)
  7. If A is invertible, then $\frac{1}{|A|} = |A^{-1}|$

- The adjont of matrix $A \in R^{n \times n}$, denoted as $\text{adj}(A)$, is the transpose of its cofactor matrix.

- E.g.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ g & h \end{vmatrix} \end{bmatrix}$$

- For any invertible $A \in \mathbb{R}^{n \times n}$, $A^{-1} = \frac{1}{|A|} \text{adj}(A)$.

Quadratic Forms and Positive Semidefinite Matrices:

- The quadratic form of a square matrix A and vector x, denoted as $x^T A x$ is:

$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

- E.g. Let $A = \begin{bmatrix} 5 & -5 \\ -5 & 1 \end{bmatrix}$, $x = (x_1, x_2, x_3)$

  The quadratic form is

$$(x_1, x_2, x_3) \begin{bmatrix} 5 & -5 \\ -5 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Note: $x^T A x = (x^T A x)^T$
  $= x^T A^T x$
  $= x^T (\frac{1}{2} A + \frac{1}{2} A^T) x$

- Let A be a symmetric matrix: $(A \in S^n)$
  1. A is positive definite (PD), denoted as $A \succ 0$ or $A > 0$, if for all non-zero vectors $x \in R^n$, $x^T A x > 0$. The set of all pd matrices is denoted as S^n_{++}.

  2. A is positive semidefinite (PSD), denoted as $A \succeq 0$ or $A \geq 0$, if $\forall$ vectors $x \in R^n$, $x^T A x \geq 0$. The set of all PSD matrices is denoted as S^n_{+}.

3. A is negative definite (ND), denoted as $A \prec 0$ or $A < 0$ if $\forall$ non-zero vectors $x \in \mathbb{R}^n$, $x^T A x < 0$.

4. A is negative semidefinite (NSD), denoted as $A \preceq 0$ or $A \leq 0$ if $\forall$ vectors $x \in \mathbb{R}^n$, $x^T A x \leq 0$.

5. A is indefinite if it is neither PSD nor NSD.
   I.e. A is indefinite if $\exists\ x_1, x_2 \in \mathbb{R}^n$ s.t.
   $x_1^T A x_1 > 0$ and $x_2^T A x_2 < 0$.

- Note:
  1. If A is PD, then $-A$ is ND and vice versa.
  2. If A is PSD, then $-A$ is NSD and vice versa.
  3. If A is indefinite, then so is $-A$.
  4. PD and ND matrices are always invertible.

- Given any matrix $A \in \mathbb{R}^{m \times n}$, the matrix $G = A^T A$, called the Gram Matrix is always PSD.

  Further, if $m \geq n$ and A is full rank, then $G = A^T A$ is PD.

Eigenvectors:

- Let $A \in R^{n \times n}$ be a square matrix. $\lambda \in C$ is an eigenvalue of A and $x \in C^n$ is an eigenvector if $Ax = \lambda x$, $x \neq 0$.

- $$\begin{aligned} Av &= \lambda v \\ &= \lambda I v \end{aligned}$$
  $$Av - \lambda I v = 0$$
  $$(A - \lambda I)v = 0$$

  If v is non-zero, then we can solve for λ using the determinant (I.e. $|A - \lambda I| = 0$)

- E.g. Let $A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$. Find the eigenvalue(s).

  Soln:
  $$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

  $$\left| \begin{bmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \right| = 0$$

  $$(-6-\lambda)(5-\lambda) - 12 = 0$$
  $$-30 + 6\lambda - 5\lambda + \lambda^2 - 12 = 0$$
  $$\lambda^2 + \lambda - 42 = 0$$
  $$(\lambda + 7)(\lambda - 6) = 0$$

  $$\lambda = 6 \text{ or } -7$$

- Properties:
  Let A be a square matrix.

  1. $\text{tr} A = \sum_{i=1}^{n} \lambda_i$

  2. $|A| = \prod_{i=1}^{n} \lambda_i$

  3. The rank of A = num of non-zero eigenvalues.

  4. If A is invertible then $1/\lambda_i$ is an eigenvalue of A^{-1} with an associated eigenvector x_i.
     I.e. $A^{-1}x_i = (1/\lambda_i)\, x_i$.

  5. The eigenvalues of a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ are just the diagonal entries $d_1, \ldots, d_n$.

- We can write all eigenvector eqns simultaneously as $AX = X\Lambda$ where:
  $X \in R^{n \times n}$ are the eigenvectors of A
  Λ is a diagonal matrix whose entries are the eigenvalues of A.

  If the eigenvectors of A are linearly indep, then matrix X will be invertible. Then, $A = X\Lambda X^{-1}$.
  A matrix that can be written in this form is called diagonalizable.

- Let A be a symmetric matrix and $A \in S^n$.
  Then:
  1. The eigenvalues of A are real.
  2. The eigenvectors of A are orthonormal.
     (We'll replace X with U in this case.)

$$A = U \Lambda U^{-1}$$
$$= U \Lambda U^T \leftarrow \text{The inverse of an orthogonal matrix is just its transpose.}$$

Using this, we can show that the definiteness of a matrix depends entirely on the sign of its eigenvalues.

$$x^T A x = x^T U \Lambda U^T x$$
$$= y^T \Lambda y \leftarrow y = U^T x$$
$$= \sum_{i=1}^{n} \lambda_i \underbrace{y_i^2}_{\text{Always positive}}$$

If all λ_i's > 0, then the matrix is PD.
If all $\lambda_i \geq 0$, then the matrix is PSD.
If all $\lambda_i < 0$, then the matrix is ND.
If all $\lambda_i \leq 0$, then the matrix is NSD.
If some $\lambda_i \geq 0$ and other $\lambda_i < 0$, then the matrix is indefinite.

Matrix Operations:

1. Matrix Addition and Subtraction:

- We can add or subtract 2 matrices if they have the same dimensions.

- E.g.

1. $$\begin{bmatrix} 1 & 7 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 2 & 9 \end{bmatrix}$$

2. $$\begin{bmatrix} 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \end{bmatrix}$$

- For matrix addition / subtraction, you simply add or subtract the elements at the same index.

2. Matrix Multiplication:

- We can only multiply 2 matrices if the number of cols of the first matrix equals the number of rows of the second matrix. The resulting matrix has the same number of rows as the first matrix and the same number of cols as the second.

I.e. Let $A \in R^{m \times n}$, $B \in R^{n \times p}$, we can multiply them because A has n cols and B has n rows.

$C = A \cdot B$, then $C = R^{m \times p}$

C gets m rows from A and p cols from B.

- To multiply 2 matrices together, we need to dot product each row of the first matrix with each col of the second.

- E.g.

1. Multiply $\begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix}$ with $\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$

Soln:

Dot product (3, 7) with (5, 0).

$(3, 7) \cdot (5, 0) = 3 \cdot 5 + 7 \cdot 0$
$= 15$

Dot product (3, 7) with (3, 1).

$(3, 7) \cdot (3, 1) = 3 \cdot 3 + 7 \cdot 1$
$= 16$

Dot product (2, 1) with (5, 0).

$(2, 1) \cdot (5, 0) = 2 \cdot 5 + 1 \cdot 0$
$= 10$

Dot product (2, 1) with (3, 1).

$(2, 1) \cdot (3, 1) = 2 \cdot 3 + 1 \cdot 1$
$= 7$

Resulting matrix: $\begin{bmatrix} 15 & 16 \\ 10 & 7 \end{bmatrix}$

2. Multiply $\begin{bmatrix} 3 & 2 \\ 7 & 1 \\ 0 & 6 \end{bmatrix}$ with $\begin{bmatrix} 9 & 4 & 6 \\ 1 & 7 & 0 \end{bmatrix}$

Soln:

$(3, 2) \cdot (9, 1) = 3 \cdot 9 + 2 \cdot 1$
$= 29$

$(3, 2) \cdot (4, 7) = 3 \cdot 4 + 2 \cdot 7$
$= 26$

$(3, 2) \cdot (6, 0) = 3 \cdot 6 + 2 \cdot 0$
$= 18$

$(7, 1) \cdot (9, 1) = 64$

$(7, 1) \cdot (4, 7) = 35$

$(7, 1) \cdot (6, 0) = 42$

$(0, 6) \cdot (9, 1) = 6$

$(0, 6) \cdot (4, 7) = 42$

$(0, 6) \cdot (6, 0) = 0$

Resulting matrix: $\begin{bmatrix} 29 & 26 & 18 \\ 64 & 35 & 42 \\ 6 & 42 & 0 \end{bmatrix}$